Applications of QR-based algorithm to determination of Lyapunov dimension of attractor of chaotic dynamical system

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In this paper QR-based algorithm to determination of Lyapunov dimension of attractor of dynamical systems represented by system of first-order differential equations is described. The algorithm of determination of Lyapunov dimension uses Lyapunov exponents. These exponents are computed with algorithm based on orthogonal-triangular decomposition QR of respective matrix. Lyapunov dimension, as example of fractal dimension, are calculated for the attractor of classic Duffing's equation and point mass moving on curve (the twin-well potential oscillator).

Keywords: Attractor, Lyapunov exponent, Lyapunov dimension, QR-algorithm, Duffing's equation, twin-well potential oscillator

1. INTRODUCTION

In recent years, fractals have been studied or applied in almost every area of science (e.g. [10], therefore it might be somewhat surprising that is no mathematical definition of a fractal. There is not even an intuitive understanding of what is and what is not a fractal. Most would, however, probably agree that two properties are characteristic of fractals: self-similarity, and fractal dimension quantities not being an integer [3]. There are a number [7] of fractal dimension quantities, including, for example, the Hausdorff dimension, the Minkowski-Bouligand dimension, the box-counting dimension, the entropy dimension or Lyapunov dimension. The difference between them is often small. It is clear that fractal dimension is not greater than topological dimension, which is always integer.

Fractal dimension can be used as a measure of complexity or irregularity of a curve or a surface. Fractal analysis has found wide applications in areas ranging from material science, power technology, computer vision to micro-electronics. In most of these applications, the common interest is to determine the fractal dimension of the concerned objects.

In applications fractal dimension is a tool that is also widely used for the quantification of chaos. Fractals are strongly connected with chaos. In this paper, chaotic dynamics of dynamical systems are taken under consideration. During the past two decades there has been considerable interest in dynamical systems [2,5,8]. Examples of chaos have been documented among physical, chemical, and biological systems (e.g. [2, 4, 12]). A review of the application of chaos theory in mechanical, civil, electrical and chemical engineering one can find in [1].

Considering a chaotic continuous dynamical system in a phase space, we monitor the long-term evolution of an infinitesimal sphere of initial conditions in a complicated way through the attractor. In more general terms, the attractor is that set of points to which trajectories approach as the number of iterations goes to infinity. More complicated systems may have more than one attractor for a given parameter value.

Fractal dimension can be used to measure the complexity or irregularity of this attractor. In this paper algorithm to determination of Lyapunov dimension (fractal dimension) of dynamical system is presented. Dynamical system must be represented by system of first-order differential equations. To determine Lyapunov exponents and dimension from system of differential equations one have to find set of linearised differential equations. The main part of algorithm is based on orthogonal-triangular decomposition QR of respective matrix. In detail algorithm is presented in section three.

2. LYAPUNOV EXPONENTS

Lyapunov exponents, named after the Russian mathematician A.M. Lyapunov, can be used to obtain a measure of the sensitive dependence of the solution of dynamical system to the initial conditions. Lyapunov exponents are a generalisation of the eigenvalues of a fixed point [9].

2.1. Fixed points of a system of differential equations

The point $\mathbf{y}^*$ where $\mathbf{f}(\mathbf{y}^*) = \mathbf{0}$ is called a fixed point of a system of differential equations $\frac{d\mathbf{y}}{dt} = \mathbf{f}$, $\mathbf{y} \in D \subset R^n$. The fixed point $\mathbf{y}^*$ of the differential equation $\frac{d\mathbf{y}}{dt} = \mathbf{f}$ is called an attractor if there exist a neighbourhood $A \subset R^n$ of $\mathbf{y}^*$ such that $\mathbf{y}(t_0) \in A$ implies $\lim_{t \to \infty} \mathbf{y}(t) = \mathbf{y}^*$. If a fixed point $\mathbf{y}^* = \mathbf{a}$ has this property for $t \to -\infty$, then $\mathbf{y}^*$ is called a repeller.

Let $\mathbf{y}^*$ be a fixed point of $\frac{d\mathbf{y}}{dt} = \mathbf{f}$. In analysing fixed points we linearise the differential equations in the neighbourhood of the fixed point. Let us assume that $\mathbf{f}$ is analytic. Thus we have a Taylor series expansion of $\mathbf{f}$ around $\mathbf{y}^*$. Linearising means that we neglect higher-order terms. In case of $\frac{d\mathbf{y}}{dt} = \mathbf{f}$ we can write in the neighbourhood of the fixed point $\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}^*)(\mathbf{y} - \mathbf{y}^*)$ and study the linear differential equation $\frac{d\mathbf{y}}{dt} = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}^*)(\mathbf{x} - \mathbf{y}^*)$. The *n x n* matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}^*)$ is called the Jacobian matrix or functional matrix. To simplify the notation the fixed point $\mathbf{y}^*$ is shifted to the origin of the phase space by $\bar{\mathbf{y}} = \mathbf{y} - \mathbf{a}$. Thus $\frac{d\bar{\mathbf{y}}}{dt} = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{y}^*)\bar{\mathbf{y}}$. The linearised system in the neighbourhood of the fixed point is of the form (we omit the bar) $\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y}$. We assume that $\det \mathbf{A} \neq 0$, and it means we exclude the case of degenerate fixed point.

In analysing the fixed points of the linear system we first determine the eigenvalues of $\mathbf{A}$. The eigenvalues $\bar{\lambda}_i$, $i = 1, \ldots, n$, are the solution of the characteristic equation $\det(\mathbf{A} - \bar{\lambda}\mathbf{I}) = 0$. Any vector satisfying equation $(\mathbf{A} - \bar{\lambda}\mathbf{I})\mathbf{v} = 0$ is called an eigenvector and Lyapunov exponents are equal to the real parts of the eigenvalues of the fixed point.

2.2. The Lyapunov exponents of dynamical system

Considering a continuous dynamical system in an n-dimensional phase space, we monitor the long-term evolution of an infinitesimal n-sphere of initial conditions. The sphere will become an n-ellipsoid due to the locally deforming nature of the flow. The i-th one-dimensional Lyapunov exponent is then defined in terms of the length of the ellipsoidal principal axis $p_i(t)$: $\lambda_i = \lim_{t\to\infty} \log_2 \frac{p_i(t)}{p_i(0)}$, where the λ_i are ordered from the largest to the smallest. Thus the Lyapunov exponents are related to the expanding or contracting nature of different directions in a phase space. Since the orientation of the ellipsoid changes continuously as it evolves, the directions associated with a given exponent vary in a complicated way through the attractor. One cannot speak of a well-defined direction associated with a given exponent.

The fact that Lyapunov exponents measure the rate of contraction or expansion they can be used as a simple criterion to distinguish between conservative and dissipative systems. For $\sum_{i=1}^{n} \lambda_i = 0$ the volume of a solution in a phase space is conserved and in this case we have a conservative system. In dissipative systems, a phase space is contracted hence $\sum_{i=1}^{n} \lambda_i < 0$. A dynamical system has an attractor only when $\sum_{i=1}^{n} \lambda_i \le 0$. For $\sum_{i=1}^{n} \lambda_i > 0$ the system is expanding and may never reach any attractor. Attractors with positive Lyapunov exponents are called strange chaotic attractors. The solution of $\frac{d\mathbf{y}}{dt} = \mathbf{f}$ with $\mathbf{y}(t_0) = \mathbf{y}_0$ is called chaotic if at least one one-dimensional Lyapunov exponent is positive.

2.3. Fractional dimension and Kolmogorov entropy

The Lyapunov spectrum is closely related to the fractional dimension of the associated strange attractor. There are a number [7] of different fractional-dimension-like quantities, including the fractal dimension, information dimension, and the correlation exponent. The difference between them is often small. It has been conjectured by Kaplan and Yorke that the information dimension d_f is related to the Lyapunov spectrum by the equation

$$d_f = j + \frac{\sum_{i=1}^{j} \lambda_i}{\left|\lambda_{j+1}\right|} \tag{1}$$

where j is defined by the condition that $\sum_{i=1}^{j} \lambda_i > 0$ and $\sum_{i=1}^{j+1} \lambda_i < 0$. The conjectured relation between d_f (a static property of an attracting set) and Lyapunov exponents appears to be satisfied for some systems.

In a system with a few positive exponents we can estimate the Kolmogorov entropy using equation $K \le \sum_{i=1}^{j} \lambda_i$ where λ_i are ordered from largest to smallest and j is the index of the minimum positive Lyapunov exponent. We can classify dynamic systems [2] using the Kolmogorov entropy. If $K = 0$ motion is regular (periodic, quasi-periodic or steady), if $0 < K < \infty$ motion is chaotic and if $K = \infty$ - accidental.

3. DETERMINATION OF LYAPUNOV DIMENSION

A defining feature of chaotic systems is the sensitive dependence on initial conditions. This implies that after a long time the distance between nearby trajectories grows exponentially as $\exp(\lambda t)$, while such divergence is as t^n for regular systems. In a regular system this divergence can also be exponential if the accessible region of phase space is unbounded. If the region is bounded locally, this exponential separation can only occur during a short period of time. To characterize these phenomena one can define the maximum Lyapunov exponent as

$$\lambda_1 = \lim_{t \to \infty} \lambda_1^n \tag{2}$$

$$\lambda_1^n = \frac{\sum_{i=1}^{n-1} \ln \frac{d_{i+1}}{d_i}}{t_n - t_1}\ , \tag{3}$$

where d_i is the distance between two points on nearby trajectories at time t_i and d_{i+1} is the distance between the same two points at time t_{i+1} obtained in a numerical integration of a differential system of equations. One must be aware that it is not possible to rigorously take a limit by numerical means, and one may find a trajectory for which λ_1^n tends to zero, as far as the computer numerical precision is concerned, while actually $\lambda_1 > 0$.

To determine a complete spectrum from a set of differential equations we recall that Lyapunov exponents are defined by the long-term evolution of the axes of an infinitesimal sphere of states. A fiducial trajectory, the centre of the sphere, is defined by the action of the nonlinear equations of motion on some initial condition. Trajectories of points on the surface of the sphere are defined by the action of the linearised equations of motion on points infinitesimally separated from the fiducial trajectory. The principal axes are defined by the evolution via the linearised equations of an initially orthonormal vector frame anchored to the fiducial trajectory.

The complete algorithm for computing complete spectrum from a set of differential equations is presented by Wolf, Swift, Swinney and Vastano [15]. This algorithm uses the Gram-Schmidt reorthonormalization (GSR) procedure on the vector frame. The range of applications of this algorithm is wide (e.g. [2,12]).

3.1. Orthogonal-triangular decomposition QR

An orthogonal matrix, or a matrix with orthonormal columns, is a real matrix whose columns all have unit length and are perpendicular to each other. If Q is orthogonal, then $\mathbf{Q}^T\mathbf{Q} = 1$. The simplest orthogonal matrices are two-dimensional coordinate rotations matrix $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$. For complex matrices, the corresponding term is unitary. Orthogonal and unitary matrices are desirable for numerical computation because they preserve length, preserve angles, and do not magnify errors. The orthogonal, or QR, factorization expresses any rectangular matrix as the product of an orthogonal or unitary matrix and an upper triangular matrix. A column permutation may also be involved: $\mathbf{A} = \mathbf{QR}$ or $\mathbf{AP} = \mathbf{QR}$ where $\mathbf{Q}$ is orthogonal or unitary, $\mathbf{R}$ is triangular, and $\mathbf{P}$ is a permutation.

Let $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, ..., \mathbf{A}^{(n)}$ denote matrixes which consists respectively of $1,2,...,n$ columns of matrix $\mathbf{A}$. Let assume that first column $\mathbf{a}_1$ of matrix $\mathbf{A}$ is not zero vector. Then

$$\mathbf{A}^{(1)} = \mathbf{Q}^{(1)}\mathbf{R}^{(1)}, \tag{4}$$

where

$$\mathbf{Q}^{(1)} = \left[\frac{\mathbf{a}_1}{\|\mathbf{a}_1\|_2}\right],\ \mathbf{R}^{(1)} = \left[\|\mathbf{a}_1\|_2\right] \text{ and } \|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{m} x_i^2}\ . \tag{5}$$

If decomposition

$$\mathbf{A}^{(i)} = \mathbf{Q}^{(i)}\mathbf{R}^{(i)} \tag{6}$$

is made then decomposition of matrix $\mathbf{A}^{(i+1)}$ on $\mathbf{Q}^{(i+1)}\mathbf{R}^{(i+1)}$ can be done by follow formulas:

$$\mathbf{Q}^{(i+1)} = \left[\mathbf{Q}^{(i)}\mathbf{q}_{i+1}\right],\ \mathbf{R}^{(i+1)} = \begin{bmatrix} \mathbf{R}^{(i)} & \mathbf{r}_{i+1} \\ 0 & s_{i+1} \end{bmatrix} \tag{7}$$

where

$$\mathbf{r}_{i+1} = (\mathbf{Q}^{(i)})^T \mathbf{a}_{i+1}, \tag{8}$$

$$\mathbf{p}_{i+1} = \mathbf{a}_{i+1} - \mathbf{Q}^{(i)}\mathbf{r}_{i+1}, \tag{9}$$

$$s_{i+1} = \|\mathbf{p}_{i+1}\|_2, \tag{10}$$

$$\mathbf{q}_{i+1} = \frac{\mathbf{p}_{i+1}}{s_{i+1}} \text{ for } s_{i+1} \neq 0. \tag{11}$$

When $s_{i+1} = 0$ then $\mathbf{Q}^{(i+1)} = \mathbf{Q}^{(i)}$. Above formulas we applied for $i = 1,2,...,n-1$. After last step for $i = n-1$ we get: unitary matrix $\mathbf{Q}^{(n)}$ (size $m \times n$) and upper triangular matrix $\mathbf{R}^{(n)}$.

3.2. QR-based algorithm to determination of Lyapunov spectrum

The algorithm employed in this paper for determining Lyapunov exponents (Lyapunov spectrum) is according to the theoretical bases of algorithm presented by Eckmann and Ruelle [6]. For first-order systems we do not have to employ QR factorization. One can use algorithm with reorthonormalization procedure [15].

To determine Lyapunov exponents from nth-order system of differential equations $\frac{d\mathbf{y}}{dt} = \mathbf{f}$ we have to find set of linearised differential equations. We can write this set in the form $\frac{d\mathbf{Y}}{dt} = \mathbf{JY}$, where $\mathbf{J} = \{J_{ij}\} = \left\{\frac{\partial f_i}{\partial y_j}\right\}$ is Jacobian matrix of size $n \times n$ and $\mathbf{Y} = \{Y_{ij}\}$ is matrix of n^2 new variables. Let denote $\mathbf{F} = \mathbf{JY}$ and write $\frac{d\mathbf{Y}}{dt} = \mathbf{F}$. Next we have to write this matrix differential equation in vector form $\frac{d\bar{\mathbf{y}}}{dt} = \bar{\mathbf{f}}$, where $\bar{\mathbf{y}}$ and $\bar{\mathbf{f}}$ are respectively $\mathbf{Y}$ and $\mathbf{F}$ matrix written row by row as n^2 elements vector.

We have system of $n+n^2$ differential equations, which we can write in form $\frac{d}{dt}\begin{bmatrix}\mathbf{y}\\ \bar{\bar{\mathbf{y}}}\end{bmatrix}=\begin{bmatrix}\mathbf{f}\\ \bar{\bar{\mathbf{f}}}\end{bmatrix}$. Initial conditions for linearised differential equations are equal to $n \times n$ identity matrix $\mathbf{I}$ written in vector form $\bar{\mathbf{i}}$.

QR-based algorithm of determination of Lyapunov exponents and Lyapunov dimension

Step 1. Set variables: $InitialTime$, $TimeStep$, $StepNumber$.
Step 2. Set variables: $T1 = InitialTime$, $T2 = T1 + TimeStep$, $M = 0$, $\mathbf{S} = \mathbf{0}$.
Step 3. Set: initial conditions for system of $n+n^2$ equations $IC = [\mathbf{y}_0, \bar{\mathbf{i}}]^T$.
Step 4. *While* $M < StepNumber$ *do* Step 5-11

Step 5. $M = M + 1$

Step 6. Find solution of system of equations with IC for $T2$: $[\mathbf{y}_M, \overline{\mathbf{y}_M}]^T$.

(Make matrix $\mathbf{A}$ with Step 7)

Step 7. *For* i *from* 1 *to* n *do*:

Step 7a: $w = n + 1 + (i-1)n$.

Step 7b: *For* j *from* 1 *to* n *do*: $A(j,i) = \overline{y_M}(w + j - 1)$.

Step 8. Make QR-decomposition: $\mathbf{A} = \mathbf{QR}$.

(Calculate Lyapunov exponents)

Step 9. *For* i *from* 1 *to* n *do*:

Step 9a. $S(i) = S(i) + \log(|R(i,i)|)$.

Step 9b. $\lambda(i) = S(i)/(T2 - InitialTime)$.

(Calculate fractal dimension - Lyapunov dimension)

Step 10. $d_f = j + \frac{\sum_{i=1}^{j}\lambda_i}{|\lambda_{j+1}|}$, $\sum_{i=1}^{j}\lambda_i > 0$, $\sum_{i=1}^{j+1}\lambda_i < 0$.

Step 11. Update: initial conditions $IC = [\mathbf{y}_M, \bar{\mathbf{i}}]^T$, $T1 = T2$, $T2 = T2 + TimeStep$.

4. EXAMPLES OF APPLICATION

4.1. Classic Duffing's equation

The sinusoidally driven twin-well oscillator has become a classic model for analysis of inherently non-linear phenomena. It is phenomena in which enormously complex chaotic motion and highly regular periodic behaviour can coexist. The equation was originally studied by Moon since 1979 [11] and was derived as mathematical model of a bucked beam or a plasma oscillations.

The simplest case for this equation is a particle placed in a twin-well potential with the base vibrating with periodic motion. When the amplitude of excitation is large enough the particle escapes from one of the potential wells and can jump from one well to the other in a randomlike, irregular manner.

The various bifurcations of steady states of the twin-well potential oscillators were invastigated in paper [13]. To be more precisely, the classic Duffing' s equation [14]

$$\frac{d^2x}{dt^2} + h\frac{dx}{dt} + \beta x^3 = F\cos(\omega t) \tag{12}$$

where $h > 0$, $F > 0$, $\beta > 0$ and $T = 2\pi/\omega$ was taken under consideration in this paper.

The Duffing's equation does not describe steady state of the twin-well potential oscillators. The correct differential equation is presented in the next subsection, where problem of a point mass moving in a vertical plane along a two-dimensional curve under the influence of vertical gravity is described.

4.2. Point mass moving on curve

In this section we shall consider a point mass moving in a vertical plane along a path described by function $y = y(x)$ under the influence of vertical gravity. For planar motion we can write

$$T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} \tag{13}$$

$$V = mgy \tag{14}$$

where T and V are kinetic and potential energies, respectively, and $\mathbf{v}$ is the velocity of the mass m. The velocity vector can be written as

$$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \tag{15}$$

with magnitude given by $v = \sqrt{\dot{x}^2 + \dot{y}^2}$. But we also have constraint $y = y(x)$ and since

$$\dot{y} = \frac{dy}{dx}\dot{x} = y_x\dot{x} \tag{16}$$

we can write the kinetic energy as

$$T = \frac{1}{2}m\left(1 + y_x^2\right)\dot{x}^2 . \tag{17}$$

A simple applications of Lagrange's equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 , \tag{18}$$

$$L = T - V \tag{19}$$

we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0 \tag{20}$$

The undamped (conservative) equation of motion of mass moving in a vertical plane along a path described by $y = y(x)$ and under the influence of vertical gravity we can write as

$$m(1 + y_x^2)\ddot{x} + m y_x y_{xx}\dot{x}^2 + mgy_x = 0 \tag{21}$$

or after mathematical manipulations as

$$\ddot{x} + \frac{y_x y_{xx}}{1 + y_x^2}\dot{x}^2 + \frac{gy_x}{1 + y_x^2} = 0 . \tag{22}$$

If we also add the effect of moving the curve horizontally by a prescribed amount function $u(t)$, the generalized velocity vector expression now is given by

$$\mathbf{v} = (\dot{x} + \dot{u})\mathbf{i} + \dot{y}\mathbf{j} = (\dot{x} + \dot{u})\mathbf{i} + y_x \dot{x}\mathbf{j}. \tag{23}$$

After augmentic the kinetic energy and applying Lagrange's equation the undamped equation of motion we can write as

$$m(1 + y_x^2)\ddot{x} + m y_x y_{xx} \dot{x}^2 + m g y_x = -m\ddot{u} \tag{24}$$

or

$$\ddot{x} + \frac{y_x y_{xx}}{1 + y_x^2}\dot{x}^2 + \frac{g y_x}{1 + y_x^2} = \frac{-\ddot{u}}{1 + y_x^2}. \tag{25}$$

5. NUMERICAL RESULTS

In this chapter the numerical results of calculations are presented for: the classic Duffing's equation and point mass moving on curve with the base vibrating with periodic motion (as an example of the twin-well potential oscillator). Base is vibrating with periodic motion described by sinusoidal function (ω is frequency and F is amplitude of function).

The automatic step size Runge-Kutta-Fehlberg integration method was used in all numerical integration of the systems of first-order differential equations in this paper. Automatic step size Runge-Kutta algorithms take larger steps where the solution is changing more slowly and uses the 4th and 5th order pair for higher accuracy. Lyapunov exponent and dimensions were computed according to algorithms described above.

5.1. Lyapunov dimension of attractor of Duffing's equation

In the first case well-known the Duffing's equation

$$\frac{d^2 x}{dt^2} + h\frac{dx}{dt} + \beta x^3 = F\cos(\omega t) \tag{26}$$

where $h > 0$, $F > 0$, $\beta > 0$ and $T = 2\pi/\omega$ can be taken under consideration as a non-autonomous system:

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -hv - \beta x^3 + F\cos(\omega t)\,. \tag{27}$$

Next, we can transfer non-autonomous system in autonomous system using substituting $z = t$ and for Duffing's equation we have:

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -hv - \beta x^3 + F\cos(\omega z) \tag{28}$$

$$\frac{dz}{dt} = 1.$$

The Jacobian matrix of right sight of autonomous system of first-order ordinary differential equations is of the form:

$$\mathbf{J}=\begin{bmatrix} 0 & 1 & 0 \\ -3\beta x^2 & -h & -F\omega\sin(\omega z) \\ 0 & 0 & 0 \end{bmatrix}. \tag{29}$$

Table 1 presents results of numerical calculations for Duffing's equation (26) or (28). The values of Lyapunov exponents (λ_i) and dimension (d_f) are presented for seven sets of parameters of equation (h, β, F, ω) and initial conditions of system of equation (28). It is shown, for example, that for $\omega = 1$ Lyapunov dimension increase when amplitude F increase. When $\omega = 0.5$ the situation is opposite. Lyapunov dimension decrease when amplitude F increase. For all cases $\sum_{i=1}^{n} \lambda_i < 0$, so the Duffing's equation with parameters considered in this subsection is dissipative system (a phase space is contracted).

It is clear to see (Fig. 1.) that at the beginning of computing the values of Lyapunov exponents are changing very rapidly and after a long period of time they are stabilized. One positive Lyapunov exponent exists in Lyapunov spectrum in all cases and it means that chaos exists in the considered examples of the Duffing's equation.

Table 1. Values of Lyapunov exponents (λ_i), Lyapunov dimension (d_f) for classic Duffing' s equation (26) with initial conditions IC .

h	β	F	ω	IC	λ_1	λ_2	λ_3	d_f
0.1	1	13	1	(10,10,0)	0.1717	0	-0.2717	2.6319
0.1	1	10	1	(0,0,0)	0.10049	0	-0.20049	2.5012
0.1	1	11	1	(0,0,0)	0.11267	0	-0.21267	2.5298
0.1	1	11	0.5	(0,0,0)	0.054206	0	-0.15421	2.3515
0.1	1	15	0.5	(0,0,0)	0.042212	0	-0.14221	2.2968
0.1	0.8	15	0.5	(0,0,0)	0.050678	0	-0.15068	2.3363
0.1	0.5	15	0.5	(0,0,0)	0.056464	0	-0.15646	2.3609

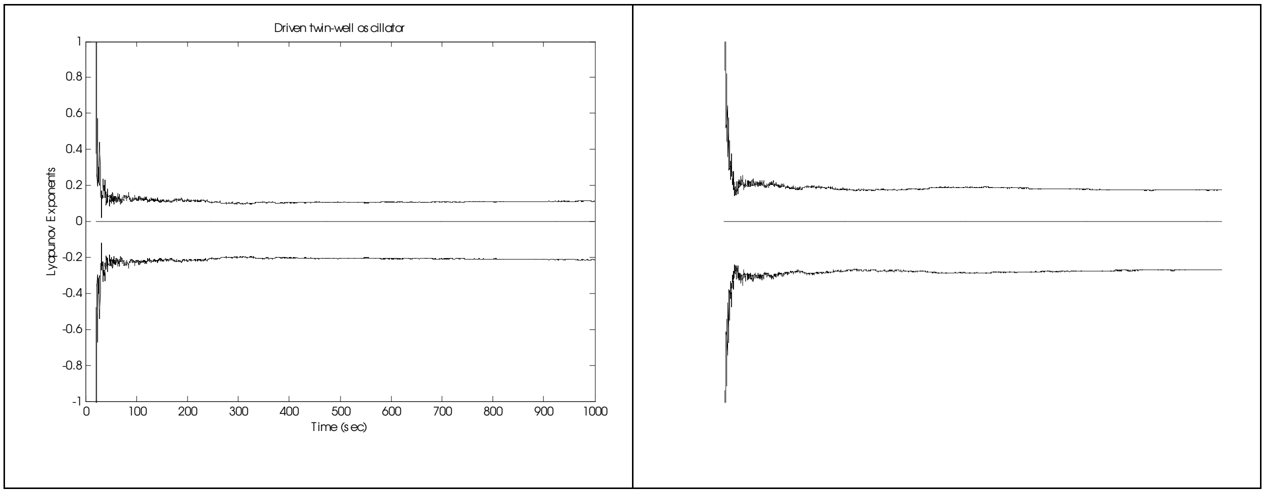

Fig. 1. Lyapunov exponents versus time for classic Duffing's equation:
(left) $h = 0.1$, $\beta = 1$, $F = 11$, $\omega = 1$, $\lambda_1 = 0.11267$, $\lambda_2 = 0$, $\lambda_3 = -0.21267$, $d_f = 2.5298$;
(right) $h = 0.1$, $\beta = 1$, $F = 13$, $\omega = 1$, $\lambda_1 = 0.1717$, $\lambda_2 = 0$, $\lambda_3 = -0.2717$, $d_f = 2.6319$.

5.2. Lyapunov dimension of attractor of the twin-well potential oscillator

In the second case (point mass moving on curve) we have a second-order ordinary differential equations, but it remains to be seen how closely it mimics Duffing's equation. For this example we can assume:

- A harmonic motion of the curve itself and of the form $u(t) = f \cdot \cos(\omega t + \varphi)$.
- The specific shape for the curve given by $y(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$.

This form of shape of curve gives two symmetric local minima at $x = \sqrt{\frac{a}{b}}$ and local maximum at $x = 0$. We can also observe that parameters of second-order differential equation of motion of point mass on curve

$$\ddot{x} + y_1(x)\dot{x}^2 + y_2(x) = \frac{-\ddot{u}}{1 + y_x^2} \tag{30}$$

are not constants for all x, where $y_1(x) = \frac{y_x y_{xx}}{1 + y_x^2}$ and $y_2(x) = \frac{g y_x}{1 + y_x^2}$ (see Fig. 2).

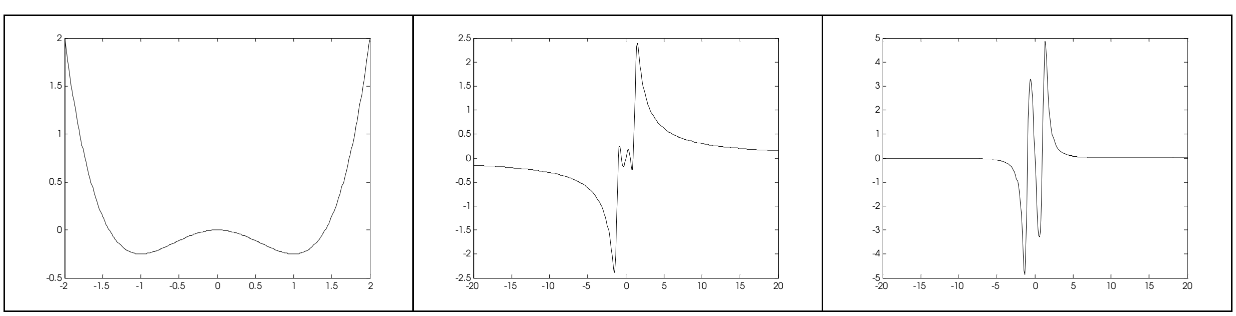

Fig. 2. Shape of curve $y(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$ (left) and parameters of differential equation: $y_1(x) = \frac{y_x y_{xx}}{1+y_x^2}$ (central) and $y_2(x) = \frac{gy_x}{1+y_x^2}$ (right) for $a = 1$, $b = 1$, $g = 9{,}81$.

In this paper equation of driven twin-well oscillator with constant parameters is taken under consideration:

$$\frac{d^2x}{dt^2} + h\left(\frac{dx}{dt}\right)^2 + \beta x^3 = F\cos(\omega t), \tag{31}$$

where $h > 0$, $\alpha > 0$, $\beta > 0$ and $T = 2\pi/\omega$.

We can write this equation as autonomous system of first-order differential equations:

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -hv^2 - \beta x^3 + F\cos(\omega z) \\ \frac{dz}{dt} &= 1. \end{aligned} \tag{32}$$

The Jacobian matrix of right sight of autonomous system of ordinary differential equations is of the form:

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 \\ -3\beta x^2 & -2hv & -F\omega\sin(\omega z) \\ 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

Table 2 presents results of numerical calculations for point-mass moving on curve with base vibrating with periodic motion. The motion of point on curve is described by equation (31) or system of equations (32). It is shown that values of Lyapunov dimension presented for four sets of parameters of equation are greater than Lyapunov dimension for Duffing's equation. It means that equation (31) is more chaotic than Duffing's equation (26). Absolute values of Lyapunov exponents are smaller. One of three exponents is zero and remaining two of them differs with certain accuracy only sign. For all cases $\sum_{i=1}^{n}\lambda_i \approx 0$, so the volume of a solution in a phase space can be regarded as conserved. In this case we have a conservative system.

Figure 3 presents Lyapunov exponents versus time for point mass moving on vibrating curve ($y(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$ with $a = 1$, $b = 1$, $g = 9{,}81$). The values of Lyapunov exponents for time 0 to 200 second are discarded.

Table 2. Values of Lyapunov exponents (λ_i), Lyapunov dimension (d_f) for point mass moving on curve (31) with initial conditions IC.

h	β	F	ω	IC	λ_1	λ_2	λ_3	d_f
0.1	1	13	1	$(0,0,0)$	0.084649	0	-0.084871	2.9974
0.1	1	11	1	$(0,0,0)$	0.11942	0	-0.1194	2.9999
0.1	1	15	0.5	$(0,0,0)$	0.056619	0	-0.055214	3.0254
0.01	1	15	1	$(0,0,0)$	0.12291	0	-0.12298	2.9994

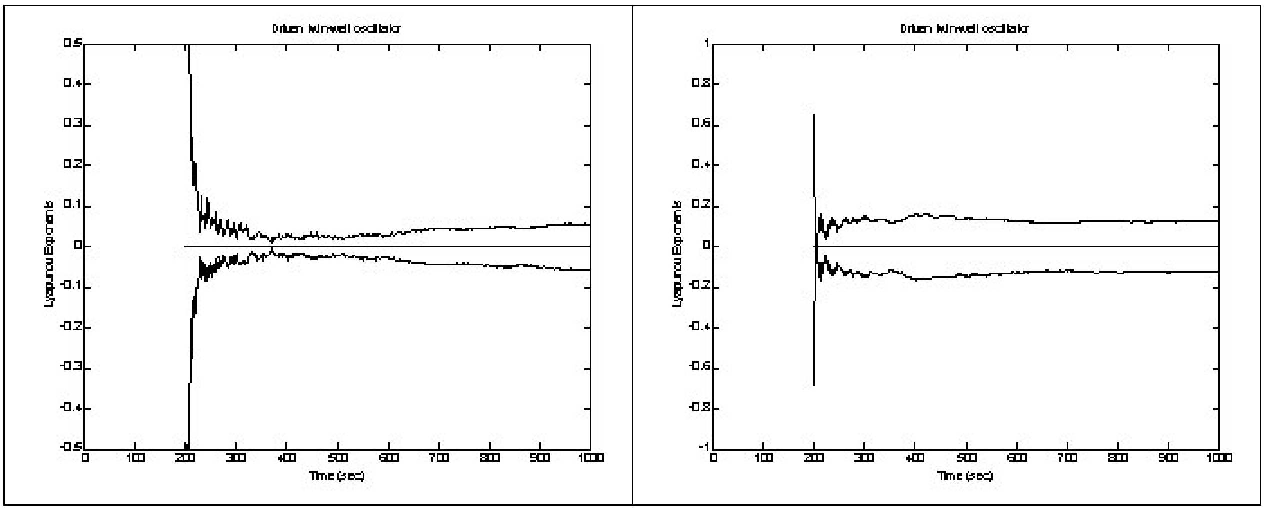

Fig. 3. Lyapunov exponents versus time for point mass moving on curve ($y(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$ with $a = 1$, $b = 1$, $g = 9{,}81$):
(left) $h = 0.1$, $\beta = 1$, $F = 15$, $\omega = 0.5$, $\lambda_1 = 0.056619$, $\lambda_2 = 0$, $\lambda_3 = -0.055214$, $d_f = 3.0254$;
(right) $h = 0.01$, $\beta = 1$, $F = 15$, $\omega = 1$, $\lambda_1 = 0.12291$, $\lambda_2 = 0$, $\lambda_3 = -0.12298$, $d_f = 2.9994$.

6. SUMMARY AND CONCLUSIONS

Lyapunov dimension (fractal dimension) can be used as a measure of complexity or irregularity of a curve. In applications fractal dimension is a tool that is also widely used for the quantification of chaos. In this paper chaotic dynamics of dynamical systems was taken under consideration.

One of the most popular algorithm for computing complete spectrum from a set of differential equations is presented in [15]. This algorithm uses the Gram-Schmidt reorthonormalization procedure on the vector frame. Complete algorithm of determination of Lyapunov exponents and dimension of dynamical system, described by system of first-order ordinary differential equations, based on orthogonal-triangular decomposition QR of respective matrix, is presented in section three.

In section five the numerical results of calculations for the classic Duffing's equation (26) and for point mass moving on curve (31) are shown. Second one is an example of the twin-well potential oscillator. In some papers such problem of the twin-well potential oscillator is investigated by Duffing's equation. As it was shown in section four Duffing's equation does not describe the twin-well potential oscillator. The numerical values of Lyapunov exponents for the same parameters are different for the Duffing's equation and the twin-well oscillator. It is clear that we can not investigate the twin-well oscillator problem using the Duffing's equation.

For cases considered in previous section we observe that dynamical system associated with point-mass moving on vibrating curve is a conservative system. It means that the volume of a solution in a phase space can be regarded as conserved. On the contrary the Duffing's equation is dissipative system (a phase space is contracted).

The second-order differential equation associated with twin-well oscillator is more complicated than the Duffing's equation. It causes problems with numerical solutions of this dynamical system. The problems are caused by second power of first different of function of position which appear in second term of equation (31).

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